FINITE C^∞-ACTIONS ARE DESCRIBED BY ONE VECTOR FIELD

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ABSTRACT. In this work one shows that given a connected C^∞-manifold M of dimension ≥ 2 and a finite subgroup $G \subset \mathrm{Diff}(M)$, there exists a complete vector field X on M such that its automorphism group equals $G \times \mathbb{R}$ where the factor $\mathbb{R}$ comes from the flow of X.

1. Introduction

This work fits within the framework of the so called *Inverse Galois Problem*: working in a category $\mathcal{C}$ and given a group G, decide whether or not there exists an object X in $\mathcal{C}$ such that $Aut_{\mathcal{C}}(X) \cong G$.

This metaproblem has been addressed by researchers in a wide range of situations from Algebra [2] and Combinatorics [4], to Topology [3]. In the setting of Differential Geometry, Kojima shows that any finite group occurs as $\pi_0(\mathrm{Diff}(M))$ for some closed 3-manifold M [8, Corollary page 297], and more recently Belolipetsky and Lubotzky [1] have proven that for every $m \geq 2$, every finite group is realized as the full isometry group of some compact hyperbolic m-manifold, so extending previous results of Kojima [8] and Greenberg [5].

Here we consider automorphisms of vector fields. Although it is obvious that the automorphism group of a vector field is never finite, we show that a given finite group of diffeomorphisms can be determined by a vector field. More precisely:

Theorem. *Consider a connected C^∞ manifold M of dimension $m \geq 2$ and a finite subgroup G of diffeomorphisms of M. Then there exists a complete G-invariant vector field X on M, such that the map*

$$\begin{array}{rcl} G \times \mathbb{R} & \to & \mathrm{Aut}(X) \\ (g,t) & \mapsto & g \circ \Phi_t \end{array}$$

is a group isomorphism, where Φ and $\mathrm{Aut}(X)$ denote the flow and the group of automorphisms of X respectively.

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Recall that, for any $m \geq 2$, every finite group G is a quotient of the fundamental group of some compact, connected C^∞-manifold M' of dimension m. Therefore G can be regarded as the group of desk transformations of a connected covering $\pi : M \to M'$ and $G \leq \mathrm{Diff}(M)$. Consequently the result above solves the Galois Inverse Problem for vector fields. Thus:

Corollary 1. *Let G be a finite group and $m \geq 2$, then there exists a connected C^∞-manifold M of dimension m and a vector field X on M such that $\pi_0(\mathrm{Aut}(X)) \cong G$.*

Our results fit into the C^∞ setting, but it seems interesting to study the same problem for other kind of manifolds and, among them, the topological ones. Namely: given a finite group $\tilde{G}$ of homeomorphisms of a connected topological manifold $\tilde{M}$ prove, or disprove, the existence of a continuous action $\tilde{\Phi}\colon \mathbb{R} \times \tilde{M} \to \tilde{M}$ such that:

(1) $\tilde{\Phi}_t \circ g = g \circ \tilde{\Phi}_t$ for any $g \in \tilde{G}$ and $t \in \mathbb{R}$.
(2) If f is a homeomorphism of $\tilde{M}$ and $\tilde{\Phi}_s \circ f = f \circ \tilde{\Phi}_s$ for every $s \in \mathbb{R}$, then $f = g \circ \tilde{\Phi}_t$ for some $g \in \tilde{G}$ and $t \in \mathbb{R}$ that are unique.

This work, reasonably self-contained, consists of five sections, the first one being the present Introduction. The others are organized as follows. In Section 2 some general definitions and classical results are given. Section 3 is devoted to the main result of this work (Theorem 1) and its proof. The extension of Theorem 1 to manifolds with non-empty boundary is addressed in Section 4. The manuscript ends with an Appendix where a technical result needed in Section 4 is proven.

For the general questions on Differential Geometry the reader is referred to [7] and for those on Differential Topology to [6].

2. Preliminary notions

Henceforth all structures and objects considered are real C^∞ and manifolds without boundary, unless another thing is stated. Given a vector field Z on a m-manifold M the group of automorphisms of Z, namely $\mathrm{Aut}(Z)$, is the subgroup of diffeomorphisms of M that preserve Z, that is

$$\mathrm{Aut}(Z) = \{f \in \mathrm{Diff}(M) : f_*(Z(p)) = Z(f(p)) \text{ for all } p \in M\}.$$

On the other hand, recall that a *regular trajectory* is the trace of a non-constant maximal integral curve. Thus any regular trajectory is oriented by the time in the obvious way and,

if it is not periodic, its points are completely ordered. As usual, a *singular trajectory* is a singular point of Z.

If $Z(p) = 0$ and Z' is another vector field defined around p then $[Z', Z](p)$ only depends on $Z'(p)$; thus the formula $Z'(p) \to [Z', Z](p)$ defines an endomorphism of T_pM called *the linear part of Z at p*. For the purpose of this work, we will say that $p \in M$ is a *source* (respectively a *sink*) of Z if $Z(p) = 0$ and its linear part at p is the product of a positive (negative) real number by the identity on T_pM.

A point $q \in M$ is called a *rivet* if

(a) q is an isolated singularity of Z,

(b) around q one has $Z = \psi\tilde{Z}$ where ψ is a function and $\tilde{Z}$ a vector field with $\tilde{Z}(q) \neq 0$.

Note that by (b), a rivet is the ω-limit of exactly one regular trajectory, the α-limit of another one and an isolated singularity of index zero.

Consider a singularity p of Z; let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of the linear part of Z at p and $\mu_1, \dots, \mu_k$ the same eigenvalues but only taking each of them into account once regardless of its multiplicity. Assume that $\mu_1, \dots, \mu_k$ are rationally independent; then $\lambda_j - \sum_{\ell=1}^m i_\ell \lambda_\ell \neq 0$ for any $j = 1, \dots, m$ and any non-negative integers $i_1, \dots, i_m$ with $\sum_{\ell=1}^m i_\ell \geq 2$, and a theorem of linearization by Sternberg (see [10] and [9]) shows the existence of coordinates $(x_1, \dots, x_m)$ such that $p \equiv 0$ and $Z = \sum_{j=1}^m \lambda_j x_j \partial/\partial x_j$. That is the case of sources ($\lambda_1 = \dots = \lambda_m > 0$) and sinks ($\lambda_1 = \dots = \lambda_m < 0$).

By definition, the *outset (or unstable manifold) R_p of a source p* will be the set of all points $q \in M$ such that the α-limit of its Z-trajectory equals p. One has:

Proposition 1. *Let p be a source of a complete vector field Z. Then R_p is open and there exists a diffeomorphism from R_p to $\mathbb{R}^m$ that sends p to the origin and Z to $a\sum_{j=1}^m x_j\partial/\partial x_j$ for some $a \in \mathbb{R}^+$. In other words, there exist coordinates $(x_1, \dots, x_m)$, whose domain R_p is identified to $\mathbb{R}^m$, such that $p \equiv 0$ and $Z = a\sum_{j=1}^m x_j\partial/\partial x_j$, $a \in \mathbb{R}^+$.*

Indeed, let Φ_t be the flow of Z; consider coordinates $(y_1, \dots, y_m)$ such that $p \equiv 0$ and $Z = a\sum_{j=1}^m y_j\partial/\partial y_j$. Up to dilation and with the obvious identifications, one may suppose that S^{m-1} is included in the domain of these coordinates. Then $R_p = \{\Phi_t(y) \mid t \in \mathbb{R}, y \in S^{m-1}\} \cup \{0\}$ and it suffices to send the origin to the origin and each $\Phi_t(y)$ to $e^{at}y$ for constructing the required diffeomorphism.

Remark 1. Observe that $R_p \cap R_q = \emptyset$ when p and q are different sources of Z.

Given a regular trajectory τ of Z with α-limit a source p, by the *linear α-limit of τ* one means the (open and starting at the origin) half-line in the vector space T_pM that is the limit, when $q \in \tau$ tends to p, of the half-line in T_qM spanned by $Z(q)$. From the local model around p follows the existence of this limit; moreover if Z is multiplied by a positive function the linear α-limit does not change.

By definition, a *chain* of Z is a finite and ordered sequence of two or more different regular trajectories, each of them called a *link*, such that:

(a) The α-limit of the first link is a source.
(b) The ω-limit of the last link is not a rivet.
(c) Between two consecutive links the ω-limit of the first one equals the α-limit of the second one. Moreover this set consists in a rivet.

The *order of a chain* is the number of its links and its *α-limit* and *linear α-limit* those of its first link.

For sake of simplicity, here countable includes the finite case as well. One says that a subset Q of M *does not exceed dimension ℓ*, or it *can be enclosed in dimension ℓ*, if there exists a countable collection $\{N_\lambda\}_{\lambda \in L}$ of submanifolds of M, all of them of dimension $\leq \ell$, such that $Q \subset \bigcup_{\lambda \in L} N_\lambda$. Note that the countable union of sets whose dimension do not exceed dimension ℓ does not exceed dimension ℓ too. On the other hand, if $\ell < m$ then Q has measure zero so empty interior.

Given a m-dimensional real vector space V, a family $\mathcal{L} = \{L_1, \ldots, L_s\}$, $s \geq m$, of half-lines of V is named *in general position* if any subfamily of $\mathcal{L}$ with m elements spans V.

Now consider a finite group $H \subset GL(V)$ of order k. A family $\mathcal{L}$ of half-lines of V is named a *control family with respect to H* if:

(a) $h(L) \in \mathcal{L}$ for any $h \in H$ and $L \in \mathcal{L}$.
(b) There exists a family $\mathcal{L}'$ of $\mathcal{L}$ with $km+1$ elements, which is in general position, such that $H \cdot \mathcal{L}' = \{h(L) \mid h \in H, L \in \mathcal{L}'\}$ equals $\mathcal{L}$.

Lemma 1. *Let $\mathcal{L}$ be a control family with respect to H and φ an element of $GL(V)$. If φ sends each orbit of the action of H on $\mathcal{L}$ into itself, then $\varphi = ah$ for some $a \in \mathbb{R}^+$ and $h \in H$.*

Indeed, as for every $L \in \mathcal{L}'$ there is $h' \in H$ such that $\varphi(L) = h'(L)$, there exist a subfamily $\mathcal{L}'' = \{L_1, \ldots, L_{m+1}\}$ of $\mathcal{L}'$ and a $h \in H$ such that $\varphi(L_j) = h(L_j)$, $j = 1, \ldots, m+1$. Therefore $h^{-1} \circ \varphi$ sends L_j into L_j, $j = 1, \ldots, m+1$, and because $\mathcal{L}''$ is in general position $h^{-1} \circ \varphi$ has to be a multiple of the identity. Since every L_j is a half-line this multiple is positive.

3. The main result

This section is devoted to prove the following result on finite groups of diffeomorphisms of a connected manifold.

Theorem 1. *Consider a connected manifold M of dimension $m \geq 2$ and a finite group $G \subset \mathrm{Diff}(M)$. Then there exists a complete vector field X on M, which is G-invariant, such that the map*

$$(g, t) \in G \times \mathbb{R} \to g \circ \Phi_t \in \mathrm{Aut}(X)$$

is a group isomorphism, where Φ denotes the flow of X.

Consider a Morse function $\mu\colon M \to \mathbb{R}$ that is G-invariant, proper and non-negative, whose existence is assured by a result of Wasserman (see the remark of page 150 and the proof of Corollary 4.10 of [11]). Denote by C the set of its critical points, which is closed, discrete (that is without accumulation points in M) so countable. As M is paracompact, there exists a locally finite family $\{A_p\}_{p \in C}$ of disjoint open set such that $p \in A_p$ for every $p \in C$.

Lemma 2. *There exists a G-invariant Riemannian metric $\tilde{g}$ on M such that if $J(p)\colon T_pM \to T_pM$, $p \in C$, is defined by $H(\mu)(p)(v, w) = \tilde{g}(p)(J(p)v, w)$, where $H(\mu)(p)$ is the hessian of μ at p, then:*

1. *If p is a maximum or a minimum then $J(p)$ is a multiple of the identity.*
2. *If p is a saddle, that is $H(\mu)(p)$ is not definite, then the eigenvalues of $J(p)$ avoiding repetitions due to the multiplicity are rationally independent.*

Proof. We start constructing a 'good' scalar product on each T_pM, $p \in C$. If p is a minimum [respectively maximum] one takes $H(\mu)(p)$ [respectively $-H(\mu)(p)$]. When p is a saddle consider a scalar product $\langle\,,\,\rangle$ on T_pM invariant by the linear action of the isotropy group G_p of G at p. In this case as $J(p)$ is G_p-invariant (of course here $J(p)$ is defined with respect to $\langle\,,\,\rangle$), $T_pM = \oplus_{j=1}^{k} E_j$ and $J(p)_{|E_j} = a_j Id_{|E_j}$ where each E_j is G_p-invariant, $a_j \neq 0$, $\langle E_j, E_\ell \rangle = 0$ and $a_j \neq a_\ell$ if $j \neq \ell$.

Besides one may suppose $a_1, \dots, a_k$ rationally independent by taking, if necessary, a new scalar product $\langle\,,\,\rangle'$ such that $\langle E_j, E_\ell\rangle' = 0$ when $j \neq \ell$ and $\langle\,,\,\rangle'_{|E_j} = b_j\langle\,,\,\rangle_{|E_j}$ for suitable scalars $b_1, \dots, b_k$.

In turns this family of scalar products on $\{T_pM\}_{p\in C}$ can be construct G-invariant. Indeed, this is obvious for maxima and minima since μ is G-invariant. On the other hand, if $C' \subset C$ is a G-orbit consisting of saddles take a point p in C', endow T_pM with a 'good' scalar product and extend to C' by means of the action of G.

It is easily seen, through the family $\{A_p\}_{p\in C}$, that of all these scalar products on $\{T_pM\}_{p\in C}$ extend to a Riemannian metric $\tilde{g}$ on M. Finally, if $\tilde{g}$ is not G-invariant consider $\sum_{g\in G} g^*(\tilde{g})$. $\square$

Let Y be the gradient vector field of μ with respect to some Riemannian metric $\tilde{g}$ as in Lemma 2. We will assume that Y is complete by multiplying, if necessary, $\tilde{g}$ by a suitable G-invariant positive function (more exactly by $e^{(Y\cdot\rho)^2}$ where ρ is a G-invariant proper function). Since μ is non-negative and proper, the α-limit of any regular trajectory of Y is a local minimum or a saddle of μ, whereas its ω-limit is empty, a local maximum or a saddle of μ.

Now $Y^{-1}(0) = C$ and, by the Sternberg's Theorem, around each $p \in C$ (note that the linear part of Y at p equals $J(p)\colon T_pM \to T_pM$ defined in Lemma 2) there exist a natural $1 \leq k \leq m-1$ and coordinates $(x_1, \dots, x_m)$ such that $p \equiv 0$ and $Y = \sum_{j=1}^m \lambda_j x_j \partial/\partial x_j$ where $\lambda_1, \dots, \lambda_k > 0$ and $\lambda_{k+1}, \dots, \lambda_m < 0$, or $Y = a\sum_{j=1}^m x_j\partial/\partial x_j$ where $a > 0$ if p is a source (that is a minimum of μ) and $a < 0$ if p is a sink (a maximum of μ.)

Let I be the set of local minima of μ, that is the set of sources of Y, and S_i, $i \in I$, the outset of i relative to Y. Obviously G acts on the set I.

Lemma 3. *In M the family $\{S_i\}_{i\in I}$ is locally finite and the set $\bigcup_{i\in I} S_i$ dense.*

Proof. First notice that $\mu(S_i)$ is low bounded by $\mu(i)$. But I is a discrete set and μ a non-negative proper Morse function, so in every compact set $\mu^{-1}((-\infty, a])$ there are only a finite number of elements of I. Therefore $\mu^{-1}((-\infty, a])$ and of course $\mu^{-1}(-\infty, a)$ only intersect a finite number of S_i. Finally, observe that $M = \bigcup_{a\in\mathbb{R}} \mu^{-1}(-\infty, a)$.

If the α-limit of the Y-trajectory of q is a saddle s, with the local model given above there exists $t \in \mathbb{Q}$ such that $\Phi_t(q)$ is close to s and $x_{k+1}(\Phi_t(q)) = \ldots = x_m(\Phi_t(q)) = 0$. Since the submanifold given by the equations $x_{k+1} = \ldots = x_m = 0$ has dimension $\leq m-1$ and $\mathbb{Q}$ and

the set of saddles are countable, it follows that the set of points coming from a saddle may be enclosed in dimension $m-1$ and its complementary, that is $\bigcup_{i\in I} S_i$, has to be dense. $\square$

The vector field Y has no rivets since all its singularities are isolated with index ± 1, therefore it has no chain; moreover the regular trajectories are not periodic.

For each $i \in I$, let $\mathcal{L}_i$ be a control family on T_iM with respect to the action of the isotropy group G_i of G at i, such that if $g(i) = i'$ then g transforms $\mathcal{L}_i$ in $\mathcal{L}_{i'}$. These families can be constructed as follows: for every orbit of the action of G on I choose a point i and $k_i m + 1$ different half-lines in general position, where k_i is the order of G_i; now G_i-saturate this first family for giving rise to $\mathcal{L}_i$. For other points i' in the same orbit choose $g \in G$ such that $g(i) = i'$ and move $\mathcal{L}_i$ to i' by means of g.

Let $\mathcal{L}$ be the set of all elements of $\mathcal{L}_i$, $i \in I$. By Proposition 1 each element of $\mathcal{L}$ is the linear α-limit of just one trajectory of Y; let $\mathcal{T}$ be the set of such trajectories. Clearly G acts on $\mathcal{T}$, since Y and $\mathcal{L}$ are G-invariant, and the set of orbits of this action is countable. Therefore this last one can be regarded as a family $\{P_n\}_{n\in\mathbb{N}'}$ where $\mathbb{N}' \subset \mathbb{N} - \{0, 1\}$, each P_n is a G-orbit and $P_n \neq P_{n'}$ if $n \neq n'$.

In turns, in each $T \in P_n$ one may choose $n-1$ different points in such a way that if $T' = g(T)$ then g sends the points considered in T to those of T'. Denoted by W_n the set of all points chosen in the trajectories of P_n.

Since $\{S_i\}_{i\in I}$ is locally finite (Lemma 3), the set $W = \bigcup_{n\in\mathbb{N}'} W_n$ is discrete, countable, closed and G-invariant. Therefore there exists a G-invariant function $\psi : M \to \mathbb{R}$, which is non negative and bounded, such that $\psi^{-1}(0) = W$. Set $Y = \varphi Z$. One has:

(a) G is a subgroup of $\mathrm{Aut}(X)$.

(b) $X^{-1}(0) = Y^{-1}(0) \cup W$, the rivets of X are just the points of W and X has no periodic regular trajectories.

(c) X and Y have the same sources, sinks and saddles. Moreover if R_i , $i \in I$, is the X-outset of i , then $R_i \subset S_i$ and $\bigcup_{i\in I}(S_i - R_i) \subset \bigcup_{T\in P_n, n\in\mathbb{N}'} T$, so $\{R_i\}_{i\in I}$ is locally finite and $\bigcup_{i\in I} R_i$ is dense.

(d) Let C_T, $T \in P_n$, $n \in \mathbb{N}'$, be the family of X-trajectories of $T - W$ endowed with the order induced by that of T as Y-trajectory. Then C_T is a chain of X of order n whose rivets are the points of $T \cap W$ and whose α-limit and linear α-limit are those of T. Besides C_T, $T \in P_n$, are the only chain of X of order n.

As each P_n is a G-orbit in $\mathcal{T}$, the group G acts on the set of chains of X and every $\{C_T \mid T \in P_n\}$ is an orbit. Thus G acts transitively on the set of α-limit and on that of linear α-limit of the chains C_T, $T \in P_n$. Recall that:

Lemma 4. *Any map $\varphi : \mathbb{R}^k \to \mathbb{R}^s$ such that $\varphi(ay) = a\varphi(y)$, for all $(a, y) \in \mathbb{R}^+ \times \mathbb{R}^k$, is linear.*

Remark 2. As it is well known, the foregoing lemma does not hold for continuous maps (in this work maps are C^∞ unless another thing is stated).

Proposition 2. *Given $f \in \mathrm{Aut}(X)$ and $i \in I$ there exists $(g, t) \in G \times \mathbb{R}$ such that $f = g \circ \Phi_t$ on R_i.*

Proof. Consider $n \in \mathbb{N}'$ such that i is the α-limit of some chain of order n. Then $f(i)$ is the α-limit of some chain of order n and there exists $g \in G$ such that $g(i) = f(i)$; therefore $(g^{-1} \circ f)(i) = i$, which reduces the problem, up to change of notation, to consider the case where $f(i) = i$.

Note that every $L \in \mathcal{L}_i$ is the linear α-limit of some $T \in \mathcal{T}$, so the linear α-limit of C_T; moreover $\mathcal{L}_i$ is the family of linear α-limit of all chains starting at i. As f sends chains starting at i into chains starting at i because f is an automorphism of X, follows that $f_*(i)$ sends $\mathcal{L}_i$ into itself.

On the other hand, since for any $T \in P_n$ one has $f(C_T) = C_{T'}$ where T' belongs to P_n as well, it has to exists $h \in G$ that sends the linear α-limit of C_T to the linear α-limit of $C_{T'}$. But both chains start at i so $h \in G_i$, which implies that $f_*(i)$ preserves each orbit of the action of G_i on $\mathcal{L}_i$. From Lemma 1 follows that $f_*(i) = ch_*(i)$ with $c > 0$ and $h \in G_i$. Therefore considering $h^{-1} \circ f$ we may suppose, up to a new change of notation, that $f_*(i) = cId$, $c > 0$.

Now Proposition 1 allows us to regard f on R_i as a map $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ that preserves the vector field $X = a\sum_{j=1}^m x_j \partial/\partial x_j$, $a \in \mathbb{R}^+$. But this last property implies that $\varphi(bx) = b\varphi(x)$ for any $b \in \mathbb{R}^+$ and $x \in \mathbb{R}^m$; therefore φ is linear (Lemma 4). Since $f_*(i) = cId$ one has $\varphi = cId$, $c > 0$; that is to say φ and $f_{|R_i}$ equal Φ_t for some $t \in \mathbb{R}$. □

Given $f \in \mathrm{Aut}(X)$, consider a family $\{(g_i, t_i)\}_{i\in}$ of elements of $G \times \mathbb{R}$ such that $f = g_i \circ \Phi_{t_i}$ on each R_i. We will show that $f = g \circ \Phi_t$ for some $g \in G$, $t \in \mathbb{R}$.

Lemma 5. *If all g_i are equal then all t_i are equal too.*

Proof. The proof reduces to the case where all $g_i = e_G$ (neutral element of G) by composing f on the left with a suitable element of G. Obviously $f = \Phi_{t_i}$ on $\overline{R}_i$.

Assume that the set of these t_i has more than one element. Fixed one of them, say t, set D_1 the union of all $\overline{R}_i$ such that $t_i = t$ and D_2 the union of all $\overline{R}_i$ such that $t_i \neq t$. Since $\{R_i\}_{i\in I}$ is locally finite and $\bigcup_{i\in I} R_i$ dense, the family $\{\overline{R}_i\}_{i\in I}$ is locally finite too and $\bigcup_{i\in I} \overline{R}_i = M$. Thus D_1 and D_2 are closed and $M = D_1 \cup D_2$. On the other hand if $p \in D_1 \cap D_2$ then $\Phi_t(p) = \Phi_{t_i}(p)$ for some $t \neq t_i$, so $\Phi_{t-t_i}(p) = p$ and $X(p) = 0$ since X has no periodic regular trajectories, which implies that $D_1 \cap D_2$ is countable. Consequently $M - D_1 \cap D_2$ is connected. But $M - D_1 \cap D_2 = (D_1 - D_1 \cap D_2) \cup (D_2 - D_1 \cap D_2)$ where the terms of this union are non-empty, disjoint and closed in $M - D_1 \cap D_2$, *contradiction.* □

Choose a $i_0 \in I$. Composing f on the left with a suitable element of G we may assume $g_{i_0} = e_G$. On the other hand, f sends each orbit of the actions of G on I into itself because the points of every orbit are just the starting points of the chains of order n for some $n \in \mathbb{N}'$. Thus f equals a permutation on each orbit of G in I and there exists $\ell > 0$ such that f^ℓ is the identity on these orbits; for instance $\ell = r!$ where r is the order of G.

Now suppose that $f^\ell = h_i \circ \Phi_{s_i}$ on R_i, $i \in I$. Then $h_i \in G_i$. Since the order of G_i divides that of G one has $f^{r\ell} = \Phi_{rs_i}$ on R_i. In short, there exists a natural number $k > 0$ such that $f^k = \Phi_{u_i}$ on R_i, and by Lemma 5 one has $f^k = \Phi_u$ on every R_i for some $u \in \mathbb{R}$.

In turns, composing f with $\Phi_{-u/k}$ we may assume, without lost of generality, that $f^k = Id$ on M.

On R_{i_0} one has $f^k = \Phi_{kt_{i_0}}$, so $t_{i_0} = 0$ and $f = Id$. But f spans a finite group of diffeomorphisms of M, which assure us that f is an isometry of some Riemannian metric $\hat{g}$ on M. Recall that isometries on connected manifolds are determined by the 1-jet at any point. Therefore from $f = Id$ on R_{i_0} follows $f = Id$ on M.

In other words the map $(g,t) \in G \times \mathbb{R} \to g \circ \Phi_t \in \mathrm{Aut}(X)$ is an epimorphism. Now the proof of Theorem 1 will be finished showing that it is an injection.

Assume that $g \circ \Phi_t = Id$ on M. As $g^r = e_G$ follows $\Phi_{rt} = Id$ whence $t = 0$ because X has no periodic regular trajectories. Thus $g = e_G$.

Remark 3. From the proof of Theorem 1 above, follows that this theorem holds for $X' = \rho X$ where $\rho\colon M \to \mathbb{R}$ is any G-invariant positive bounded function. Indeed, reason as before with $(\rho\psi)Y$ instead of ψY.

4. Actions on manifolds with boundary

Let P be an m-manifold with non-empty boundary ∂P. Set $M = P - \partial P$. First recall that there always exist a manifold $\tilde{P}$ without boundary and a function $\tilde{\varphi} : \tilde{P} \to \mathbb{R}$ such that zero is a regular value of $\tilde{\varphi}$ and P diffeomorphic to $\tilde{\varphi}^{-1}((-\infty, 0])$; so let us identify P and $\tilde{\varphi}^{-1}((-\infty, 0])$.

Now assume that G is a finite subgroup of $\mathrm{Diff}(P)$, P is connected and $m \geq 2$. Then G sends ∂P to ∂P and M to M; thus by restriction G becomes a finite subgroup of $\mathrm{Diff}(M)$.

Let X' be a vector field as in the proof of Theorem 1 with respect to M and $G \subset \mathrm{Diff}(M)$. By Proposition 3 in the Appendix (Section 5) applied to M and X', there exists a bounded function $\varphi \colon \tilde{P} \to \mathbb{R}$, which is positive on M and vanishes elsewhere, such that the vector field $\varphi X'$ on M prolongs by zero to a (differentiable) vector field on $\tilde{P}$.

Lemma 6. *For every $g \in G$ the vector field X_g equal to $(\varphi \circ g)X'$ on M and vanishing elsewhere is differentiable.*

Proof. Obviously X_g is smooth on $\tilde{P} - \partial P$. Now consider any $p \in \partial P$. As $g \colon P \to P$ is a diffeomorphism, there exist an open neighborhood A of p on $\tilde{P}$ and a map $\hat{g} \colon A \to \tilde{P}$ such that $\hat{g} = g$ on $A \cap P$. Shrinking A allows to assume that $B = \hat{g}(A)$ is open, $\hat{g} \colon A \to B$ is a diffeomorphism and $A - \partial P$ has two connected components A_1, A_2 with $A_1 \subset M$ and $A_2 \subset \tilde{P} - P$; note that $\hat{g}(A_1) \subset M$, $\hat{g}(A_2) \subset \tilde{P} - P$ and $\hat{g}(A \cap \partial P) \subset \partial P$.

Thus $(X_g)_{|A} = \hat{g}_*^{-1}(X_\varphi)_{|B}$ since X' is G-invariant. $\square$

On P set $X = \sum_{g \in G} X_g$. Then $X_{|\partial P} = 0$ and $X_{|M} = \rho X'$ where $\rho = \sum_{g \in G}(\varphi_{|M}) \circ g$. Clearly $\rho \colon M \to \mathbb{R}$ is positive bounded and G-invariant, so by Remark 3 Theorem 1 also holds for $X_{|M}$. Moreover X is complete on P.

If $f \colon P \to P$ belongs to $\mathrm{Aut}(X)$ then $f_{|M}$ belongs to $\mathrm{Aut}(X_{|M})$ and $f = g \circ \Phi_t$ on M and by continuity on P. In other words, Theorem 1 also holds for any connected manifold P, of dimension ≥ 2, with non-empty boundary.

5. Appendix

In this appendix we prove Proposition 3 that was needed in the foregoing section. First consider a family of compact sets $\{K_r\}_{r \in \mathbb{N}}$ in an open set $A \subset \mathbb{R}^n$, such that $K_r \subset \mathring{K}_{r+1}$, $r \in \mathbb{N}$, and $\bigcup_{r \in \mathbb{N}} K_r = A$.

Lemma 7. *Given a family of positive continuous functions $\{f_r : A \to \mathbb{R}\}_{r\in\mathbb{N}}$ there exists a function $f : A \to \mathbb{R}$ vanishing on $\mathbb{R}^n - A$ and positive on A such that, whenever $r \in \mathbb{N}$, one has $f \leq f_j$, $0 \leq j \leq r$, on $A - K_r$.*

Proof. One may assume $f_0 \geq f_1 \geq \ldots \geq f_r \geq \ldots$ by taking $min\{f_0, \ldots, f_r\}$ instead of f_r if necessary. Consider functions $\varphi_r : \mathbb{R}^n \to [0,1] \subset \mathbb{R}$, $r \in \mathbb{N}$, such that each $\varphi_r^{-1}(0) = K_{r-1} \cup (\mathbb{R}^n - \overset{\circ}{K}_{r+1})$ [as usual $K_j = \emptyset$ if $j \leq -1$].

Let D be a partial derivative operator. Multiplying each f_r by some $\varepsilon_r > 0$ small enough allows to suppose, without loss of generality, $\varphi_r \leq f_r/2$ on A and $\mid D\varphi_r \mid \leq 2^{-r}$ on $\mathbb{R}^n$ for any D of order $\leq r$.

Set $f = \sum_{r\in\mathbb{N}} \varphi_r$. By the second condition on functions φ_r, whenever $\tilde{D}$ is a partial derivative operator the series $\sum_{r\in\mathbb{N}} \tilde{D}\varphi_r$ uniformly converges on $\mathbb{R}^n$, which implies that f is differentiable. On the other hand it is easily checked that $f(\mathbb{R}^n - A) = 0$, $f > 0$ on A and $f \leq f_r \leq \ldots \leq f_0$ on $A - K_r$. □

One will say that a function defined around a point p of a manifold is *flat at p* if its ∞-jet at this point vanishes. Note that given a function ψ on a manifold and a function $\tau : \mathbb{R} \to [0,1] \subset \mathbb{R}$ flat at the origin and positive on $\mathbb{R} - \{0\}$ (for instance $\tau(t) = e^{-1/t^2}$ if $t \neq 0$ and $\tau(0) = 0$), then $\tau \circ \psi$ is flat at every point of $(\tau \circ \psi)^{-1}(0) = \psi^{-1}(0)$ and $Im(\tau \circ \psi) \subset [0,1]$.

Lemma 8. *Consider an open set A of a manifold M and a function $f : A \to \mathbb{R}$. Then there exists a function $\varphi : M \to \mathbb{R}$ vanishing on $M - A$ and positive on A, such that the function $\hat{f} : M \to \mathbb{R}$ given by $\hat{f} = \varphi f$ on A and $\hat{f} = 0$ on $M - A$ is differentiable.*

Proof. The manifold M can be seen as a closed imbedded submanifold of some $\mathbb{R}^n$. Let $\pi : E \to M$ be a tubular neighborhood of M. If the result is true for $\pi^{-1}(A)$ and $f \circ \pi : \pi^{-1}(A) \to \mathbb{R}$, by restriction it is true for A and f. In other words, it suffices to consider the case of an open set A of $\mathbb{R}^n$.

We will say that a function $\psi : A \to \mathbb{R}$ is *neatly bounded* if, for each point p of the topological boundary of A and any partial derivative operator D, there exists an open neighborhood B of p such that $\mid D\psi \mid$ is bounded on $A \cap B$. First assume that f is neatly bounded. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a function that is positive on A and flat at every point of $\mathbb{R}^n - A$; then φ satisfies Lemma 8.

Indeed, only the points $p \in (\bar{A} - A)$ need to be examined. Consider an natural $1 \leq j \leq n$; since $j_p^\infty \varphi = 0$ near p one has $\varphi(x) = \sum_{i=1}^n (x_i - p_i)\tilde{\varphi}_i(x)$ and from the definition of partial derivative follows that $(\partial \hat{f}/\partial x_j)(p) = 0$. Thus $\partial \hat{f}/\partial x_j = (\partial \varphi/\partial x_j)f + \varphi \partial f/\partial x_j$ on A and $\partial \hat{f}/\partial x_j = 0$ on $\mathbb{R}^n - A$, which shows that f is C^1.

Since obviously the function $\partial f/\partial x_j$ is neatly bounded and $\partial \varphi/\partial x_j$ is flat on $\mathbb{R}^n - A$, the same argument as before applied to $(\partial \varphi/\partial x_j)f$ and $\varphi \partial f/\partial x_j$ shows that f is C^2 and, by induction, the differentiability of f.

Let us see the general case. On A the continuous functions $\mid Df \mid +1$, where D is any partial derivative operator, give rise to a countable family of continuous positive functions $g_0, \ldots, g_r, \ldots$. Let $\{K_r\}_{r \in \mathbb{N}}$ be a collection of compact sets such that $K_r \subset \overset{\circ}{K}_{r+1}$, $r \in \mathbb{N}$, and $\bigcup_{r \in \mathbb{N}} K_r = A$. By Lemma 7 there exists a function $\rho : \mathbb{R}^n \to \mathbb{R}$ vanishing on $\mathbb{R}^n - A$ and positive on A such that $\rho \leq g_j^{-1}$, $0 \leq j \leq r$, on $A - K_r$, $r \in \mathbb{N}$.

For every $k \in \mathbb{N}$ let $\lambda_k : \mathbb{R} \to \mathbb{R}$ be the function defined by $\lambda_k(t) = t^{-k}e^{-1/t}$ if $t > 0$ and $\lambda_k(t) = 0$ elsewhere. Then the function $\tilde{f} = \lambda_0(\rho/2)f$ is neatly bounded on A. Indeed, consider any $p \in (\bar{A} - A)$ and any partial derivative operator D. Then $D\tilde{f}$ equals a linear combination, with constant coefficients, of products of some partial derivatives of ρ, a function $\rho^{-k}e^{-2/\rho} = \lambda_k(\rho)e^{-1/\rho}$ and some partial derivative $D'f$. On the other hand, there always exists a natural ℓ such that $g_\ell = \mid D'f \mid +1$. But near p one has $e^{-1/\rho} \mid D'f \mid \leq \rho \mid D'f \mid \leq \rho g_\ell \leq 1$; therefore $D\tilde{f}$ is bounded close to p.

Finally, take a function $\tilde{\varphi} : \mathbb{R}^n \to \mathbb{R}$ positive on A and flat at every point of $\mathbb{R}^n - A$ and set $\varphi = \tilde{\varphi}\lambda_0(\rho/2)$. $\square$

Proposition 3. *Consider a vector field X on an open set A of a manifold M. Then there exists a bounded function $\varphi : M \to \mathbb{R}$, which is positive on A and vanishes on $M - A$, such that the vector field $\hat{X}$ on M defined by $\hat{X} = \varphi X$ on A and $\hat{X} = 0$ on $M - A$ is differentiable.*

Proof. Regard M as a closed imbedded submanifold of some $\mathbb{R}^n$; let $\pi : E \to M$ be a tubular neighborhood of M. Then there exists a vector field X' on $\pi^{-1}(A)$ such that $X' = X$ on A and, by restriction of the function, it suffices to show our result for X' and $\pi^{-1}(A)$. That is to say, we may suppose, without loss of generality, that A is an open set of $\mathbb{R}^n$.

In this case on A one has $X = \sum_{j=1}^n f_j \partial/\partial x_j$. Applying Lemma 8 to every function f_j yields a family of functions $\varphi_1, \ldots, \varphi_n$. Now it is enough setting $\varphi = \varphi_1 \cdots \varphi_n$.

Finally, if φ is not bounded take $\varphi/(\varphi + 1)$ instead of φ. $\square$

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